SEVERAL RISK MEASURES IN PORTFOLIO SELECTION: IS IT WORTHWHILE?

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Resumen

Este artículo aborda el problema de selección de activos utilizando tres medidas del riesgo ampliamente utilizadas: varianza o desviación típica, Valor en Riesgo y Valor en Riesgo condicional. Nuestro objetivo es evaluar si resolver el problema de selección de activos con varias medidas del riesgo es relevante o no, dada la complejidad computacional que supone. La principal contribución de este artículo es la solución de dos modelos que consideran varias medidas del riesgo: el modelo de media-varianza-VaR y el modelo media-VaR-CVaR. La inclusión del VaR como uno de los objetivos a minimizar convierte el problema en no convexo, por tanto el método de resolución propuesto está basado en una heurística: algoritmo genético multiobjetivo. Nuestros resultados muestran la adecuación de el enfoque multiobjetivo para resolver el problema de optimización de carteras y enfatiza la importancia de abordar los modelos de media-varianza-VaR o media-VaR-CVaR en lugar del modelo media-varianza-CVaR.

Abstract

This paper is concerned with asset allocation using a set of three widely used risk measures, which are the variance or deviation, Value at Risk and the Conditional Value at Risk. Our purpose is to evaluate whether solving the asset allocation problem under several risk measures is worthwhile or not, given the added computational complexity. The main contribution of the paper is the solution of two models that consider several risk measures: the mean-VaR-CVaR model and the mean-σ-VaR model. The inclusion of VaR as one of the objectives to minimize leads to nonconvex problems, therefore the approach we propose is based on a heuristic: multi-objective genetic algorithms. Our results show the adequacy of the multi-objective approach for the portfolio optimization problem and emphasize the importance of dealing with mean-σ-VaR or mean-VaR-CVaR models as opposed to mean-σ-CVaR.
1. Introduction

Portfolio selection is obtained maximizing expected return and minimizing risk. There are several ways of measure risk of a portfolio. The classical measure of risk is the variance or standard deviation used by seminal work of Markowitz (1952). In this case, the portfolio selection is done by solving a quadratic problem. This risk measure weights equally positive against negative returns. This is justified by assuming either that investors have quadratic utility functions or that asset returns are drawn from a multivariate elliptical distribution. Other measures known as downside risk measures have been proposed to capture the left-hand side of a return distribution which involves risks. Bawa (1975) and Fisburn (1977) introduce a general definition of downside risk in the form of lower partial moments. An example is semi-variance, which is a special case of lower partial moments.

Nowadays researchers and practitioners are focused on Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) as measures of market risk. VaR of a portfolio is the lowest amount such that the loss will not exceed it with probability $1-\alpha$. CVaR is the conditional expectation of losses above the VaR. VaR and CVaR can be used to balance risk and return. While CVaR can be efficiently minimized using linear programming and non-smoothing techniques (Rockafellar and Uryasev, 2000), minimizing VaR leads to a non-convex and non-differential risk-return problem and smoothing techniques (Gaivoronski and Pflug, 2005) or heuristic optimization techniques need to be applied (Gilli et al. 2006). Although variance is still the most widely used measure of risk in the practice of portfolio selection, VaR and CVaR are used as risk limit and to control risk by the fund management industry. This has motivated the inclusion of VaR and CVaR as constraints in the classical mean-variance (Alexander et al. 2007).

Each risk measure, variance, VaR and CVaR, captures different aspects of risk. Therefore, it could be worthy to introduce them jointly in the portfolio analysis. In the literature we can find some works that explore this idea. For instance, Roman et al. (2007) proposed the use of two risk measures (variance and CVaR) in order to perform a portfolio selection.

The mean-variance-CVaR model proposed is a multi-objective problem. A method to obtain all the efficient solutions is given and tested on the solutions of mean-variance model and mean-CVaR model. The aim of this paper is to show how different efficient portfolios are if we consider two risk measures in portfolio selection. A portfolio
selection considering more than one risk measure is conducted by using multiobjective genetic algorithms. Additionally, this paper is a novelty since risk management with percentile functions is a very important topic and new optimization algorithms for portfolio allocation are needed. The proposal is validated on real data from Eurostoxx 50 index.

The structure of the paper is as follow: Section 2 formulates the problems under one and several risk measures. Section 3 presents the multiobjective genetic algorithm proposed. Data and empirical results are presented in Section 4. Finally, in Section 5 some conclusions are drawn.

2. The portfolio selection problem under a set of risk measures

Although investors propose a different risk measure the three stages of Markowitz framework remains: the stage of security analysis, which comprises the collection of data and the estimation of relevant parameters; the stage of portfolio analysis, where the efficient set of portfolios is computed by solving an optimization problem dependant on the risk measure; and the stage of portfolio choice, which entails the selection of an optimal portfolio depending on the specific preferences of the investor. When a new risk measure, such VaR and CVaR is proposed the computational complexity of the optimization problem can increase. Hence, the problem to solve in the stage of portfolio analysis varies. While variance can be minimized using a quadratic problem and CVaR can be minimized using a linear problem following Rockafellar and Uryasev (2000), the use of VaR leads to a non-convex NP-hard optimization problem.

2.1 Portfolio selection under one risk measure

When the return distribution is normal the mean-variance model, mean-CVaR model and mean-VaR model to solve for obtaining the efficient set of portfolios are equivalent. The mean-variance model can be formulated as follows:

\[
\begin{align*}
\min_{w} & \quad w' \Omega w \\
\text{s. t.} & \quad w'r \geq r^* \\
& \quad w'1 = 1 \\
& \quad w_j \geq 0 \quad \forall j = 1, \ldots, n
\end{align*}
\]
where \( w \) is a vector containing the percentage of the budget invested in each asset, \( \Omega \) is the covariance matrix and \( r \) is a vector containing the expected rate of return of each asset. An important property of the variance of the portfolio return is that it allows defining the objective function as a quadratic function.

The mean-VaR model uses the concept of VaR as a measure of risk instead of variance. The formal definition can be expressed as follows: given a cumulative probability distribution function of returns \( F(x) \) and a risk threshold \( \alpha \) then the VaR is

\[
VaR_\alpha(w) = \inf \{ x \mid F(x) \geq \alpha \}.
\]

If \( F(x) \) is a continuous, strictly increasing function \( VaR_\alpha(w) \) is the \( \alpha \)-quantile. If \( F(w) \) is upper semi-continuous, then \( VaR_\alpha(w) \) is the smallest value of \( x \) for which \( F(X \leq x) \geq \alpha \). Then, efficient portfolios are the solution for the following problem:

\[
\begin{align*}
\text{Min} & \quad VaR_\alpha(x) = \inf \{ x \mid F(x) \geq \alpha \} \\
\text{s.t.} & \quad w'r \geq r^* \\
& \quad w'1 = 1 \\
& \quad w_j \geq 0 \quad \forall j = 1, \ldots, n
\end{align*}
\]  

[2]

VaR is difficult to optimize for discrete distributions since is non-convex and has multiple local extrema. Mostly, approaches rely on linear approximation of the portfolio risks and assume a joint normal distribution of the underlying parameters (Jorion, 1996, Duffie and Pan, 1997). Optimization requires smoothing or heuristic techniques as those presented in Gaivoronski and Pflug (2005), Gilli et al. (2006) or Alfaro et al. (2008).

On the other hand, CVaR is closely related to VaR. CVaR can be defined as the conditional expected loss under the condition that it exceeds VaR. For general distributions, CVaR is more attractive than VaR since it is sub-additive and convex. Also, it has been named coherent measure of risk in the sense of Artzner et al. (1999) and Acerbi and Tasche (2002). Hence, the problem of minimizing CVaR for finding efficient portfolios is convex. As Rockefellar and Uryasev (2000, 2002) demonstrate that, in the case of discrete random variables with \( T \) possible outcomes, it is possible to linearise the CVaR by introducing a vector of auxiliary variables. Then investors have to solve a linear problem where CVaR as objective function has been replaced by,
\[
\begin{align*}
\text{Min } CVaR_\alpha(w) &= \text{Min } -v + \frac{1}{n} \sum_{i=1}^{T} y_i \\
\text{s. t.} & \\
\sum_{j=1}^{n} -r_j w_j + v \leq y_i & \forall i = 1, \ldots, T \\
y_i \geq 0 \forall i = 1, \ldots, T \\
w' r \geq r^* \\
w' 1 = 1 \\
w_j \geq 0 \forall j = 1, \ldots, n
\end{align*}
\]

Solving these three models allows us to obtain sets of optimal portfolios. The question is how different the efficient portfolios obtained with different models are. Are portfolios that are efficient in the mean variance model (problem 1) efficient in mean-VaR or mean-CVaR space or not? If the answer is not, that is, if each risk measure reflects different risk dimensions of a portfolio then we should consider the differences between the efficient frontiers, given that increasing the number of risk measures implies defining the investor’s preferences through more complex utility functions. Finally, if there is a difference that permits to balance the risk measures considered then it could be worthwhile to formulate the selection problem as a multiobjective problem with two or more risk measures to minimize.

### 2.2 Portfolio selection under several risk measures

The inclusion of more than one risk measure to obtain efficient portfolios has been approached from two perspectives: introducing additional constrains or introducing additional objective functions as multiobjective problems. Early examples on using an additional variable for the mean-variance model can be found in Konno et al. (1993) in which mean, absolute deviation and skewness are combined. The skewness is maximized subject to constraints in mean and absolute deviation. Konno and Suzuki (1995) show an efficient algorithm to optimize a mean-variance-skewness model. Alexander et al. (2007) obtain efficient portfolios under mean-variance model when VaR or CVaR are used as constraints. A pure multiobjective proposal is found in Roman et al. (2007) where a mean-variance-CVaR model is proposed and an optimization approach is given. Introducing several risk measures allows controlling more information about portfolio risk. Under normal distributions of returns the three measures provide the same optimal portfolio (Rockafellar and Uryasev, 2000) and a multiobjective risk is worthless. However, for skewed distributions, VaR and CVaR optimal portfolios may be quite different.
A multiobjective problem in general is expressed as $Max\{f_1(w), f_2(w), \ldots, f_n(w)\}$. In the classical asset allocation problem there are two objectives to optimize: mean and variance. If an additional risk measure is introduced the concept of efficient portfolio is the same but the problem has a third dimension. A feasible solution $w'$ Pareto dominates another feasible solution $w$ if $f_i(w') \geq f_i(w)$ for all $i$ with at least one strict inequality. For instance, in a mean-VaR-CVaR model the efficient portfolios are the Pareto efficient solutions of a multiobjective problem in which the expected value is maximized while the VaR and CVaR are minimized. The problem may be formulated as:

$$Max \ (w'r, -VaR_\alpha (x), -CVaR_\alpha (x))$$

s.t.

$$w'1 = 1$$

$$w_j \geq 0 \ \forall j = 1,\ldots, n$$

Minimizing VaR does not control losses exceeding VaR since it does not take into account the shape of the tail. Then, introducing CVaR as a second measure of risk captures these losses. In this mean-VaR-CVaR model, $w'$ is preferred to portfolio $w'$ if and only if $E(R_{w'}) \geq E(R_w)$, $VaR_\alpha (w') \leq VaR_\alpha (w')$ and $CVaR_\alpha (w') \leq CVaR_\alpha (w')$ with at least one strict inequality. When we consider three objectives we obtain a surface of efficient portfolios instead of a line. One issue addressed in this paper is how the efficient frontier changes when we increase the measures of risk.

On the other hand, in a mean-variance-VaR model the efficient portfolios are the Pareto efficient solutions of:

$$Max \ (w'r, -w'\Omega w, -VaR_\alpha (x))$$

s.t.

$$w'1 = 1$$

$$w_j \geq 0 \ \forall j = 1,\ldots, n$$

This paper is related to Roman et al. (2007), which study efficient portfolios considering more than one measure of risk as objective function. The main difference of our contribution is that we present a practical optimization approach where VaR is considered as a measure of risk in a multiobjective problem. In addition, we evaluate if the inclusion of several risk measures is needed or irrelevant for efficient portfolios. As Roman et al. (2007) indicates VaR leads to complex problems, whose solution requires the development of new algorithms. We focus on mean-VaR-CVaR and mean-variance-VaR models.

By definition, the portfolio optimization problem proposed by Markowitz (1952) is multi-objective with two conflicting criteria: maximising the return and minimising the level of risk. Introducing several risk measures increases the number of objectives. Multi-objective optimization problems do not have a single solution but a set of solutions equally optimal that define the efficient frontier (or Pareto-optimal front).

Genetic algorithms (GAs) (see, Holland, 1975, Goldberg, 1989) are stochastic optimization techniques that mimic the way species evolve in nature. GAs emulate this process by encoding the points of the search space (called individuals) in a chromosome-like shape and evolving a population of them through a number of generations using mechanisms drawn from natural evolution. Along the evolution the individuals’ fitness is evaluated according to how well they solve the problem at hand. The better the fitness of an individual, the more chances it has to produce offspring for the next generation. As the generations progress, it results in the prevalence of stronger solution over weaker ones. Thus, the evolution process tends to near optimal solutions.

A GA initiates the process of searching by randomly generating an initial population of possible solutions. The performance of each solution is evaluated using a fitness function, which is a measure of how good the performance of the solution is. Then, a new generation is produced according to the three main operators of the GA: selection, crossover and mutation.

Selection determines which solutions are chosen for mating according to the principal of survival of the fittest (i.e. the better the performance of the solution, the more likely it is to be chosen for mating and therefore the more offspring it produces). In this work we used tournament selection. The tournament selection method works by choosing a group of $q$ individuals randomly from the population and selecting the best individual in terms of fitness from this group.

Crossover allows an improvement in the species in terms of the evolution of new solutions that are fitter than any seen before. The crossover operator combines the features of two parents to create new solutions. One or several crossover points are selected at random on each parent and then, complementary fractions from the two parents are spliced together to form a new chromosome, as shown in Fig. 1.
Mutation reintroduces values that might have been lost through selection or crossover, or creates totally new features. The mutation operator alters a copy of a chromosome. One or more locations are selected on the chromosome and replaced with new randomly generated values. Mutation is used to help ensure that all areas of the search space remain reachable providing higher variation in the chromosomes of each population. It also allows the reintroduction of features that might have been lost during the selection procedure.

The cycle selection-crossover-mutation-evaluation is performed until a termination criterion is met (for instance, a predetermined number of generations) (see Figure 1).

For multi-objective GA (Coello, 2006) the concept of fitness changes. The fitness of an individual it is not anymore how well it solves the problem, but it is based on the Pareto optimality concept. The fitness of an individual is a function of how many individuals it dominates and by how many individuals it is dominated. Thus, nondominated individuals have the highest possible fitness and the rest of individuals are ranked according to their dominance relations.

Other important concept that is often used in GAs is the concept of elitism. In an elitist selection technique the best individuals of the population are automatically selected to go to the next generation without undergoing crossover or mutation. In the context of multiobjective GAs, the use of a subpopulation (usually called archive) where the nondominated individuals are stored along the generations guarantees that nondominated solutions are not lost during the run and that a solution reported as nondominated is nondominated with respect to any other solution generated by our algorithm.

### 3.1 GA implementation

The GA implementation used in this work is based on ECJ\(^1\), a research evolutionary computation system in Java developed at George Mason University's Evolutionary Computation Laboratory (ECLab). For the multi-objective aspect of the optimization the SPEA2 (Strength Pareto Evolutionary Algorithm 2) package of ECJ was used (Zitzler, 2001). SPEA2 is an improved version of SPEA which incorporates a fine-grained

\(^1\) [http://cs.gmu.edu/~eclab/projects/ecj/](http://cs.gmu.edu/~eclab/projects/ecj/)
fitness assignment strategy, a density estimation technique and an enhanced archive truncation method. As most of the multi-objective evolutionary methods it keeps an archive where the non-dominated solutions are stored. The size of the archive is set by the user so that if the number of non-dominated solutions is bigger than the archive size the archive is truncated.

The algorithm works as follows:

- In step 1 and 2 the archive, \( A(g) \), where the non-dominated solutions are stored and the population, \( P(g) \), are initialized. \( A(0) \) is an empty set and \( P(0) \) is initialized at random.
- In step 3 the generation counter \( g \) is set to 1 and then the evolution loop starts.
- In step 4 and 5 the individuals in the population and the archive are evaluated.
- According to this evaluation a new archive is created in step 6 containing all the non-dominated individuals found in the union of the previous archive and the population.
- If the size of the resulting archive exceeds the archive size, in step 7 the archive is truncated. This truncation method removes those individuals which are at the minimum distance of another individual. This way the characteristics of the non-dominated front are preserved and outer solutions are not lost.
- The termination criterion in step 8 stops the algorithm when the number of generations has been completed.
- In step 9 tournament selection with replacement is performed in the archive set in order to fill the mating pool, \( M(g) \).
- The new population, \( P(g) \), is created in step 10 by applying crossover and mutation to the mating pool.
- In step 11 the generation counter is increased.

The control parameters of the GA used are quite standard. The GA is generational. It uses tournament selection with tournament size of 7. The probabilities of crossover and mutation are 1 and 0.05 respectively. The population size is 5000 and the archive size is 100. The run finishes after 100 generations. Each individual is encoded as a vector of integers ranging from 0 to 99. Every element of the vector represents the percentage of the budget invested in that particular asset \( w_j^G \leq 0 \ j=1,\ldots, n \). Therefore, the length of the vector equals the number of assets available in the portfolio. However, the summation of these weights will not be 1, violating the constraint \( \sum_{j=1}^{n} w_j = 1 \).
constraint imposes the need of normalizing the vector during the decoding process as follows:

\[ w_j = \frac{w_j^{GA}}{\sum_{j=1}^{n} w_j^{GA}} \]  

where \( w_j \) represents the weight invested in asset \( j \). However, these normalized weights are real values.

4. Data and empirical analysis

The data used in this work were extracted from the Bloomberg database. It is a set composed of fifty stocks which belonged to the Eurostoxx 50 index in January 2008. Three stocks with negative expected return in the analysis period were eliminated. We use daily data of these stocks from January 2003 to December 2007. This gives us 1300 observations per stock. We chose daily data instead of monthly data to avoid inaccurate VaR estimates from small samples.

Table 1 reports the descriptive analysis of the data identifying companies by country. It can be observed that the mean daily return is close to zero. This is consistent with computing VaR under the assumption of expected daily return equal to zero. We compute BeraJarque (BJ) statistic to test normality. The BJ statistic has a chi-squared distribution with two degrees of freedom under the null hypothesis that returns are normally distributed. As can be observed the minimum value of the BJ statistic is 59.29 while the critical value is 9.21 for a 1% significance level. Hence normality is rejected in all cases and portfolio risk level ranking is different depending on the measure selected: standard deviation, VaR or CVaR. Also, Table 1 reports standard deviation, VaR and CVaR at 95% confidence level. For example, we can find Siemens AG with 1.617% of deviation, 2.535% of VaR and 3.569% of CVaR while SAP AG has higher deviation 1.719%, lower VaR 2.43% and higher CVaR 3.798%.

In order to show the irregularity of VaR against standard deviation and CVaR, we computed the VaR, standard deviation and CVaR for all feasible portfolios composed of two stocks with high weight in the index, such as Total SA and Banco Santander SA. Figure 2 shows the convexity of the standard deviation and CVaR with a global minimum whereas VaR is not a convex function and it presents several local
minimums. Moreover, the standard deviation, VaR and CVaR are not interchangeable risk measures because the minimum deviation portfolio and the minimum VaR portfolio do not coincide. Total SA weight in the optimal portfolio using deviation is 59%, using VaR is 49% and using CVaR is 68%.

[INSERT FIGURE 2 HERE]

For testing the reliability of the multi-objective evolutionary approach (GA), we solved the classical mean-variance problem using GAs and we compared the results obtained with those of the classical Quadratic Program (QP). Also, we solved the mean-CVaR problem using GAs and we compared the results obtained with those of linear program (LP) proposed by Rockefellar and Uryasev (2000, 2002).

[INSERT FIGURE 3 HERE]
[INSERT TABLE 2 HERE]

Figure 3 illustrates that the $\sigma$-efficient frontiers calculated using GAs and QP and CVaR-efficient frontiers calculated using GAs and LP are overlapped. Table 2 reports the differences between efficient frontiers. The mean error of the $\sigma$-optimal portfolios computed using QP and GA is close to zero (0.0049%), the optimal portfolio with the highest difference is 0.0196% which further confirms the good results provided by the genetic algorithm to minimize deviation and maximize return. Similar conclusions are obtained when we maximize return and minimize CVaR and we compare these results of the linear programming applying the transformation of the objective function proposed by Rockefellar and Uryasev (2000, 2002). The mean error of the CVaR-optimal portfolios computed using LP and GA is close to zero (0.0190%), the optimal portfolio with the highest difference is 0.0430%. Also, Figure 3 illustrates the VaR-efficient frontier computed using GA. The frontier is irregular since VaR is a non-convex risk measure.

So far we have proven that multi-objective GAs can easily solve Markowitz classical problem and the mean-CVaR problem without linear transformation. We want to quantify how different these risk measures (deviation, CVaR and VaR) are in the optimal portfolios to know if risk measures are interchangeable in the subset of efficient portfolios. When we ignore a measure of risk, we are assuming that the used measure and the ignored measure are substitutive, that is, both measures give similar efficient frontier. If both measures are complementary, then they both should be in the objective function.
Table 3 reports the difference in deviation and CVaR between optimal portfolios under different risk measures. The mean difference between standard deviation of $\sigma$-optimal portfolios and the standard deviation of CVaR-optimal portfolios is 0.008% with a maximum difference of 0.0237% while the mean difference with VaR-optimal portfolios is 0.0314% with a maximum difference of 0.063%. It means that deviation is a measure more related to CVaR than VaR. With respect to CVaR, the mean CVaR difference between CVaR-optimal portfolios and the CVaR of $\sigma$-optimal portfolios is 0.0201% with a maximum difference of 0.0531% while the mean difference with VaR-optimal portfolios is 0.0895% with a maximum difference of 0.2362%. Again the results reflects more similarity between standard deviation and CVaR than between CVaR and VaR. Figure 4 illustrates that differences representing $\sigma$-optimal portfolios, CVaR-optimal portfolios and VaR-optimal portfolios in mean-deviation, mean-CVaR and mean-VaR spaces. As can be observed, even though in each space the optimal portfolios are those obtained as optimal for the risk measure that define the space, $\sigma$-optimal portfolios and CVaR-optimal portfolios are close in all spaces while VaR-optimal portfolios are more separated in all spaces. This confirms that obtaining optimal portfolios for several risk measures is not so important when the risk measures considered are standard deviation and CVaR. However, when VaR is considered as a risk measure, the inclusion of a new risk measure such as CVaR allows obtaining different efficient portfolios in terms of these risks.

Figure 5 presents the results obtained solving problems [4] and [5]. $\sigma$-optimal portfolios, CVaR-optimal portfolios and VaR-optimal portfolios are represented in the $\sigma$-CVaR, $\sigma$-VaR and VaR-CVaR spaces for different fixed returns. The differences between $\sigma$-optimal portfolios and CVaR-optimal portfolios in the $\sigma$-CVaR space are the smallest and the differences between VaR-optimal portfolios and CVaR-optimal portfolios in the VaR-CVaR space are the biggest as Table 3 reflects.

As $\sigma$-optimal portfolios are the efficient portfolios in mean-variance space they are also efficient in mean-variance-VaR or mean-variance-CVaR space. This is due to the fact that for a fixed return their volatility is the minimum. That is, other portfolios could have a smaller CVaR or VaR for the same return but not a smaller deviation. The same happens with CVaR-optimal portfolios. They are also efficient or non-dominated in mean-variance-CVaR space. Therefore, mean-variance-CVaR efficient portfolios are all the portfolios included in the area between $\sigma$-optimal portfolios and CVaR-optimal portfolios.
portfolios. As is shown in Figure 5 this surface is narrow, which means that the set of portfolios in which investors could select their portfolio selection is similar solving Markowitz (1952) model in [1], Rockafellar and Uryasev (2000) in [3] or Roman et al. (2007). However, mean-VaR-CVaR and mean-variance-VaR model offer wider surface of efficient portfolios to investors. Therefore, it is interesting to solve problems [4] and [5] since the solutions obtained offer a wide range of portfolios for the investor to choose according to his preferences. Taking into consideration the asset allocation with the three risk measures, that is, a mean-variance-VaR-CVaR model, would be similar to consider mean-VaR-CVaR or mean-variance-VaR models since variance and CVaR in the set of efficient portfolios generate a narrow surface.

Table 4 illustrates the risk values of the $\sigma$-optimal portfolio, the CVaR-optimal portfolio, the VaR-optimal portfolios, the $\sigma$-VaR optimal portfolio and the $\sigma$-CVaR optimal portfolio for two level of expected return, 0.08% and 0.1%. Minimum values of $\sigma$, VaR and CVaR values are highlighted for each return. As can be observed, for each return $\sigma$-VaR optimal portfolio has a higher deviation and lower VaR than $\sigma$-optimal portfolio and lower deviation and higher VaR than VaR-optimal portfolio. $\sigma$-VaR optimal portfolios balance deviation and VaR. Graphically all the portfolios that make up the efficient surface are located in the gap shown in Figure 5 when $\sigma$-optimal portfolios and CVaR-optimal portfolios are represented in $\sigma$-VaR space. The same happens when we analyze the VaR-CVaR efficient portfolio for each level of return proposed. They have a lower VaR and higher CVaR than CVaR-optimal portfolios and higher VaR and lower CVaR than VaR-optimal portfolios. The surface of efficient portfolios for mean-VaR-CVaR model is composed by portfolios in the gap between VaR-optimal portfolios and CVaR-optimal portfolios represented in the VaR-CVaR space in Figure 5.

5. Conclusions

Variance, VaR and CVaR are different risk measures that capture different aspects of the return distribution related to risk. Usually asset allocation has been done through a multiobjective problem where expected return is maximized and one risk measure is minimized. The inclusion of several risk measures is an issue that is being addressed recently by researchers. In this paper, we evaluate which joint combination of risk measures is more informative.

Based on our results we can conclude that investors do not need balance variance and CVaR when selecting an optimal portfolio. Optimal portfolios in the variance-CVaR
space form a narrow surface. Then, ignoring a risk dimension does not change portfolio risk values significantly.

It is more informative to solve a mean-variance-VaR model or a mean-VaR-CVaR model since variance and CVaR or VaR and CVaR behave differently and, as a consequence the resulting surface of efficient portfolios is significantly wider. Investors can choose their optimal portfolio minimizing the maximum loss with a confidence level, that is VaR, for an expected return required and variance level ranging between the variance of the mean-variance efficient portfolio and the mean-VaR efficient portfolio. Alternatively and with similar results, investors can choose their optimal portfolio minimizing the maximum loss with a confidence level, that is VaR, for an expected return required and mean loss under the VaR level ranging between the CVaR of the mean-CVaR efficient portfolio and the mean-VaR efficient portfolio.

Finally, the results obtained have shown the feasibility of using multiobjective genetic algorithms for this problem. The multiobjective genetic algorithm has dealt with the non-convexity characteristics of VaR and it has solved the three-objectives models efficiently.
References


Figure 1. Flow Chart of a Genetic Algorithm.

- Generate Initial Population
  - Decoding
  - Encoding
- Evaluate Initial Population
- Select Individuals for Matting
- Cross New Population
- Mutate New Population
  - Decoding
  - Encoding
- Evaluate New Population
  - No
  - End?
  - Yes
Table 1. Summary of data statistics.

<table>
<thead>
<tr>
<th>Country</th>
<th>Company</th>
<th>Mean</th>
<th>SD</th>
<th>VaR95%</th>
<th>CVaR95%</th>
<th>BJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>France</td>
<td>Air Liquide</td>
<td>0.048</td>
<td>1.270</td>
<td>1.928</td>
<td>2.800</td>
<td>175.6</td>
</tr>
<tr>
<td>France</td>
<td>Alcatel-Lucent</td>
<td>0.004</td>
<td>2.333</td>
<td>3.472</td>
<td>5.457</td>
<td>640.5</td>
</tr>
<tr>
<td>Germany</td>
<td>Allianz SE</td>
<td>0.039</td>
<td>1.853</td>
<td>2.911</td>
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<td>VaR (%)</td>
<td>CVaR (%)</td>
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Mean; the average daily return, SD; standard deviation, VaR_{95%}; the 95% 1-day VaR, and CVaR_{95%}; the 95% 1-day CVaR are expressed in percentage and are computed with daily returns from January 2003 to December 2007. BJ is the Bera-Jarque statistic.

**Figure 2. Deviation, VaR and CVaR for portfolios with two assets.**

Portfolios are composed by Total SA and Banco Santander SA.
**Figure 3.** Mean-variance, Mean-CVaR and Mean-VaR efficient frontiers computed with different techniques.

Mean-variance efficient frontier has been computed using multiobjective GA and quadratic programming, QP. Mean-CVaR efficient frontier has been computed using multiobjective GA and linear programming, LP. Mean-VaR efficient frontier has been computed using multiobjective GA.
### Table 2. Differences in deviation and CVaR between optimal portfolios calculated using different optimization techniques.

<table>
<thead>
<tr>
<th></th>
<th>$w_{\alpha}^{GA} - w_{\sigma}^{QP}$</th>
<th>$w_{\text{CVaR}}^{GA} - w_{\text{CVaR}}^{LP}$</th>
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<tbody>
<tr>
<td>Mean</td>
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<td>0.0190</td>
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<td>0.0014</td>
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<tr>
<td>Max</td>
<td>0.0196</td>
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</table>

Note: values expressed in percentage.

### Table 3. Differences in deviation and CVaR between optimal portfolios under different risk measures.

<table>
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<tr>
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<th>SD differences</th>
<th>CvaR differences</th>
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<tr>
<td></td>
<td>$w_\sigma-w_{\text{CVaR}}$</td>
<td>$w_\sigma-w_{\text{VaR}}$</td>
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<tr>
<td>Max</td>
<td>0.0237</td>
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</table>

Note: differences in deviation and CvaR are expressed in percentage. $w_\sigma$, $w_{\text{CVaR}}$, and $w_{\text{VaR}}$ represent optimal portfolios under deviation, CVaR and VaR risk measures, respectively.
Figure 4. Mean-variance, Mean-CVaR and Mean-VaR efficient frontiers represented in Mean-variance, Mean-CVaR and Mean-VaR spaces.

SD line represents $\sigma$-optimal portfolios, VaR line represents VaR-optimal portfolios and CVaR line represents CVaR-optimal portfolios.
Figure 5. $\sigma$-optimal portfolios, CVaR-optimal portfolios and VaR-optimal portfolios represented in the $\sigma$-CVaR, $\sigma$-VaR and VaR-CVaR spaces.
Table 4. Risk attributes of efficient portfolios in several spaces

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<th>Min σ</th>
<th>SD</th>
<th>CVaR</th>
<th>VaR</th>
<th>Min CVaR</th>
<th>SD</th>
<th>CVaR</th>
<th>VaR</th>
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* indicates minimum value for given return